

## Exploring Cavalieri's Principle

### Introduction

Cavalieri's principle was first introduced in Bonaventura Cavalieri's book, *Geometria Indivisibilibus* in 1635. Cavalieri was an Italian mathematician who, along with others of his time, invented and used methods to solve area and volume problems. His principle essentially states that "If, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal."<sup>1</sup> When I first saw his principle, I thought the it seems quite obvious and understandable when I, for example, visualized a perfectly stacked penny and a stack of penny that is crooked with the same height, as seen in Figure 1.



Figure 1

Although the shapes of the two figures are dissimilar, the volume of both solids and the areas of a horizontal cross section at a given height are all the same because the radius of each penny is constant and the pennies of the crooked figure are just shifted around. Nothing jumped out at me.

I, however, became more interested and intrigued in Cavalieri's principle as my math teacher began to compare the volume and horizontal cross sections of a hemisphere and of a solid that is a cylinder minus a cone, as shown in Figure 2:

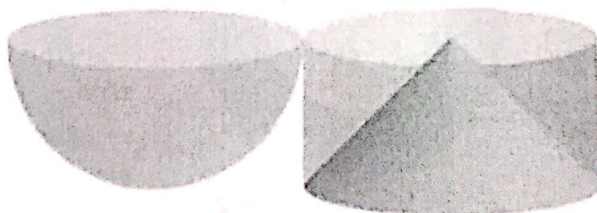


Figure 2

It became apparent that these two solids indeed have the same volume and the areas of the cross sections of the two at any given height are the same,

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<sup>1</sup> Weisstein, Eric W. "Cavalieri's Principle." MathWorld--A Wolfram Web Resource. Wolfram Research, Inc. 16 Nov. 2013  
<<http://mathworld.wolfram.com/CavalierisPrinciple.html>>.

thus satisfying the principle. I was surprised when I was first exposed to how the principle is applied to the solids above, with cross sections as shown in Figure 3.



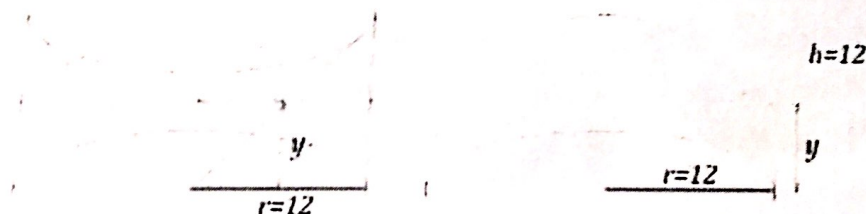
Figure 3

It didn't visually occur to me that the shaded areas of the two shapes would be the same, let alone the volumes of the two shapes! This shocking discovery, as well as the simplicity and elegance of Cavalieri's principle, fascinated me and drove me to dive deeper into his principle.

In this exploration, I attempted to show how Cavalieri's principle works with the two aforementioned solids. In order to accomplish this, I took at least three cross sections of both bodies at the same distance from their respective bases and calculated their areas to see if they are equal. If the two solids satisfy those premises, then I would be able to conclude that the volume of the two are congruent and can verify the theory by calculating the volume of the two 3D shapes. Many suggested that Cavalieri stated a principle for volumes of solids that anticipated the integral calculus; thus, I would be using mathematical skills such as integral calculus, Pythagorean theorem, and algebra in this exploration.

### Equal Cross Section Areas

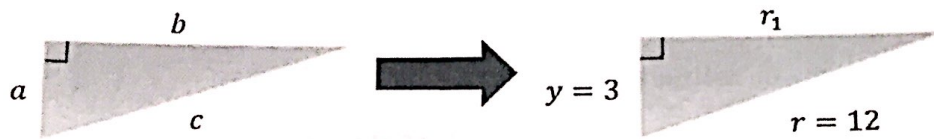
First, I needed to show that Cavalieri's principle is true by calculating the areas of several cross sections of the two different solids. I imagined a hemisphere and a cylinder minus a cone with radii  $r=12$  and equal heights of  $h=12$ , like so:



The first cross section would be at  $y=3$ . In order to calculate the area of the cross section taken from the hemisphere, I needed to use the circle area formula,  $\pi r^2$ . However, I realized that I had to find the radius of the circle at  $y=3$  first. I drew a right triangle with the radius of the hemisphere ( $r$ ) as the hypotenuse, the height of the cross section from its base ( $y$ ) and the radius of the cross section ( $r_1$ ) as the legs. Because any point of the hemisphere is equal



distance away from its center, the radius of the hemisphere can become the hypotenuse. Though the Pythagorean theorem is usually referred to as  $c^2 = a^2 + b^2$ , I will replace the variables to those that I am using in this exploration:



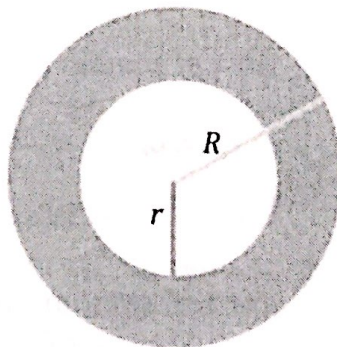
Now, using the Pythagorean theorem, I found the radius of the small cross section:

$$\begin{aligned} r^2 &= y^2 + r_1^2 \\ 12^2 &= 3^2 + r_1^2 \\ r_1^2 &= 12^2 - 3^2 \\ r_1^2 &= 135 \\ r_1 &= \sqrt{135} \end{aligned}$$

As for calculating the area of circle, in which  $r$  is the radius of the cross section circle:

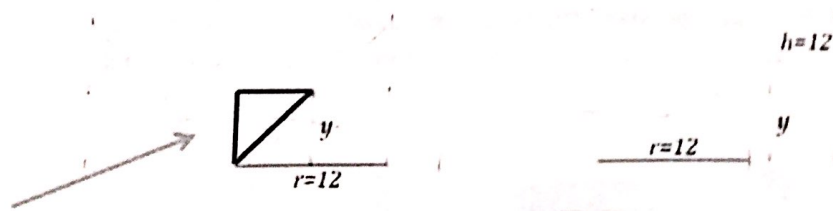
$$\begin{aligned} \text{area of a circle} &= \pi r_1^2 \\ &= \pi(\sqrt{135})^2 \\ &= 135\pi \end{aligned}$$

In order to compare the areas, I also calculated the cross section area of the cylinder minus a cone at the same height. Because the area I wanted to calculate was an annulus, "a region between two concentric circles which have different radii,"<sup>2</sup> which looks like this:



I had to subtract the area of the smaller circle from the area of the bigger circle. The radius of the cylinder is  $r = 12$  as it is constant at any height. However, I was slightly confused when finding the radius of the smaller circle,  $r_2$ , because the radius of the cone varies at different heights from its base. When looking at the following diagram, I realized I was able to draw a triangle that would help me find the radius, like so:

<sup>2</sup> Simmons, Bruce. "Mathwords: Annulus." *Mathwords: Annulus*. Web. 25 Oct. 2013. <<http://www.mathwords.com/a/annulus.htm>>.



Turned out, the triangle that I drew was a 45-45-90 Isosceles triangle; therefore,  $r_2 = y$ . Because I've already established that  $y = 3$ , by substitution,  $r_2 = 3$ . Now that I can calculate the area of the annulus:

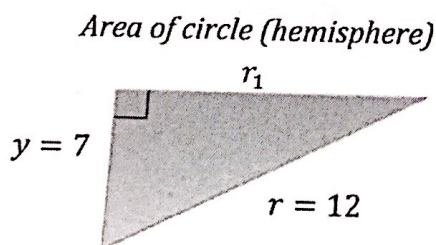
$$\begin{aligned} \text{Area of larger circle} &= \pi r^2 \\ &= \pi(12)^2 \\ &= 144\pi \end{aligned}$$

$$\begin{aligned} \text{Area of smaller circle} &= \pi r_2^2 \\ &= \pi(3)^2 \\ &= 9\pi \end{aligned}$$

$$\begin{aligned} \text{Area of the annulus} &= 144\pi - 9\pi \\ &= 135\pi \end{aligned}$$

The two cross section areas both equal  $135\pi$ ! At first glance the two cross sections did not seem to equal each other for me; yet, after this calculation, there are clear evidence that suggest otherwise, which is quite amazing. Just to be sure that the areas are equal at a given height, I found the area of the cross sections at two other different heights.

With the same radii, the height of the cross sections away from its bases was now 7. This time, I did the calculations side-by-side to show that they did equal each other, and they were equal again!



$$\begin{aligned} r^2 &= y^2 + r_1^2 \\ 12^2 &= 7^2 + r_1^2 \\ r_1^2 &= 12^2 - 7^2 \\ r_1^2 &= 135 \\ r_1 &= \sqrt{95} \end{aligned}$$

$$\begin{aligned} \text{Area of a circle} &= \pi r_1^2 \\ &= \pi(\sqrt{95})^2 \\ &= 95\pi \end{aligned}$$

Area of annulus (cylinder-cone)

$$\begin{aligned} \text{Area of larger circle} &= \pi r^2 \\ &= \pi(12)^2 \\ &= 144\pi \end{aligned}$$

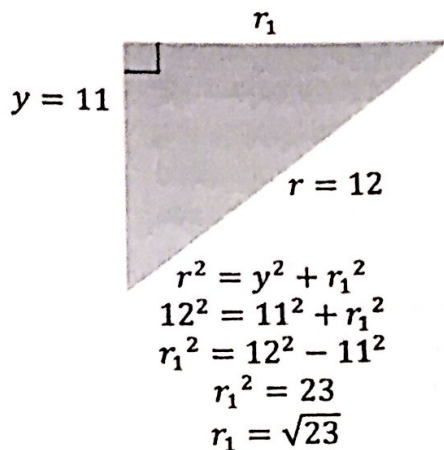
$$\begin{aligned} \text{Area of smaller circle} &= \pi r_2^2 \\ &= \pi(7)^2 \\ &= 49\pi \end{aligned}$$

$$\begin{aligned} \text{Area of annulus} &= 144\pi - 49\pi \\ &= 95\pi \end{aligned}$$

I performed the calculations again but with a different height,  $y = 11$ :



Area of circle (hemisphere)



$$\begin{aligned} \text{Area of a circle} &= \pi r_1^2 \\ &= \pi(\sqrt{23})^2 \\ &= 23\pi \end{aligned}$$

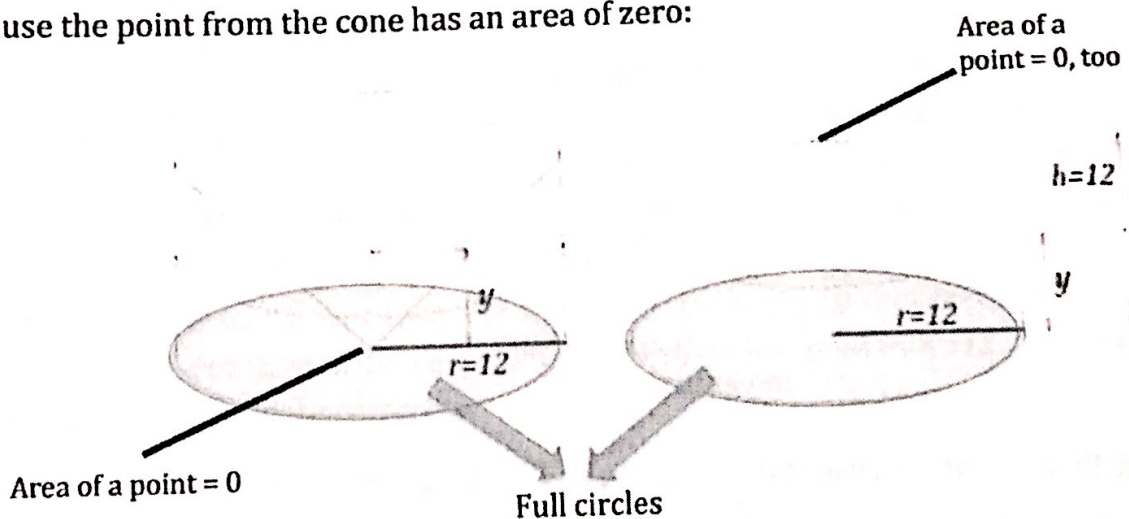
Area of annulus (cylinder-cone)

$$\begin{aligned} \text{Area of larger circle} &= \pi r^2 \\ &= \pi(12)^2 \\ &= 144\pi \end{aligned}$$

$$\begin{aligned} \text{Area of smaller circle} &= \pi r_2^2 \\ &= \pi(11)^2 \\ &= 121\pi \end{aligned}$$

$$\begin{aligned} \text{Area of annulus} &= 144\pi - 121\pi \\ &= 23\pi \end{aligned}$$

I chose not to calculate and compare the top and the bottom slices of the two solids since the bottom of both shapes are all full circles. I was able to discover this, as the base of the hemisphere was a full circle. The base of the cylinder with a point from the bottom of the cone removed was, also, a full circle because the point from the cone has an area of zero:



The top of both solids also equals each other because the area of the hemisphere at the very top is simply a point, which contains an area of zero. Similarly, the top area of the cylinder minus a cone turned out to be zero as well since it is a full circle from the cylinder minus a full circle from the top of the cone!

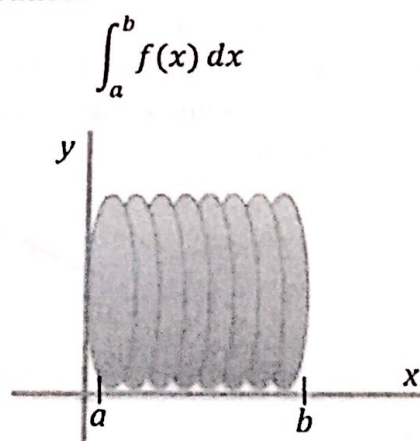
So far, these two solids agreed with the first part of the Cavalieri's principle, that "the sections made by planes parallel to and at the same distance from their respective bases are always equal." However, I was aware that in

order to complete the full proof of Cavalieri's principle, I also needed to demonstrate the second part of his principle!

### Equal Volume

The second half of his principle affirms "the volumes of the two solids are equal" if the two solids' cross sections at the same distance from their bases are equal. Since I thought this was the more complex part of his principle, I consulted my math teacher and I discovered that because all the 'slices' of the solids have different areas due to the constantly changing radius, I couldn't just add up all the slices to obtain a volume. As a result, I needed to employ integral calculus to find the volume. I, again, consulted my teacher and researched, on my own, how to do integration when I only had knowledge on the basics such as differentiation and limits because I wanted to try to fully understand how calculus applies to Cavalieri's principle and how he foresaw the development of calculus.

After learning how to work with integrals, I decided I would continue to use the previous solids with the same measurements, radii and height to find the volume of the two solids. At first, I found this generic definite integral formula and with the following illustration:



Visual representation of definite integration

Looking at this visual representation of what  $\int_a^b f(x) dx$  does, I suddenly realized that this was exactly what Cavalieri was trying to convey through his principle of each parallel slices needs to have the same area! These equal area slices stack up horizontally, similar to how each cross section stack up and has the same area in Cavalieri's principle. This finally gave me insight into the connection between the principle and integrals that I read about when I researched.

I knew to use definite integral, in which I add up a period or length of infinitely small pieces, and not indefinite integral since I'm adding up the slices from height zero,  $a = 0$ , to the top,  $b = 12$ .



While I was previously slightly unsure of what  $f(x)$  function in the formula needed to be, after the looking at the diagram, it's now clear that the function needs to calculate the each slices of the solid.

At first, in an attempt to calculate the volume of the hemisphere with calculus, I tried to see if this equation would calculate the volume since the are of each the area of each slice in a hemisphere is  $\pi r^2$ :

$$\int_0^{12} \pi r^2 dx$$

But wait! I suddenly realized that this integral equation, with the function  $\pi r^2$ , wouldn't work because it didn't take into account the varying radius that changes as the height of the hemisphere increases. The fact that I was differentiating with respect to  $x$  also wouldn't produce a formula that matches the geometric formula for calculating volume of a hemisphere. I was genuinely confused at this point.

Through trial and error and looking back at what I did in the previous section, I found that I needed a function that takes into account the changing radius in different slices of the hemisphere. I referred back to the calculations I did, whereby I used the Pythagorean theorem to derive the radius of the cross section:

$$r^2 = y^2 + r_1^2$$

In order to find the new radius, it was simply a matter of rearranging the Pythagorean theorem so that  $r_1^2 = r^2 - y^2$ . Now looking at the equation for calculating the area of the cross section circle for the hemisphere,  $\pi r_1^2$ , I can substitute  $r_1^2$  in the equation for  $r^2 - y^2$ . This way, the new equation for calculating the area of a circle,  $\pi(r^2 - y^2)$ , would take into account and calculate each radius of the cross section at a given distance from its base. I also realized what  $dx$  does is that it stacks up infinitely thin pieces of the function in a direction from 0 to 12. Because the height of the solid is represented by  $y$ , I needed to differentiate with respect to  $y$  from when  $y=0$  to  $y=12$ . The new equation, I discovered, becomes:

$$\int_0^{12} (\pi(r^2 - y^2)) dy$$

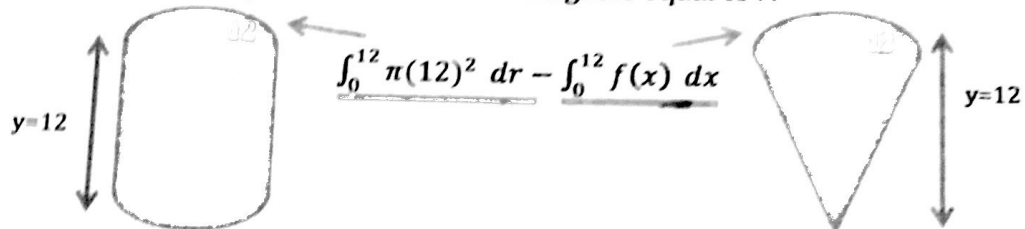
And since  $r = 12$ :

$$\begin{aligned} & \int_0^{12} (\pi((12)^2 - y^2)) dy \\ &= 144\pi - \pi y^2 \\ &= 144\pi y - \frac{\pi}{3} y^3 \\ &= \left(144\pi(12) - \frac{\pi}{3}(12^3)\right) - \left(144\pi(0) - \frac{\pi}{3}(0^3)\right) \\ &= \left(1728\pi - \frac{1728\pi}{3}\right) - 0 \end{aligned}$$

$$= (1728\pi - 576\pi)$$

$$= 1152\pi$$

According to this integration, the volume of the hemisphere is  $1152\pi$ . Next, I tried to use integrals to calculate the volume of the cylinder minus cone solid by subtracting the volume of the cone from the volume of the cylinder. I differentiated with respect to  $r$  because the height is equal to  $r$ :



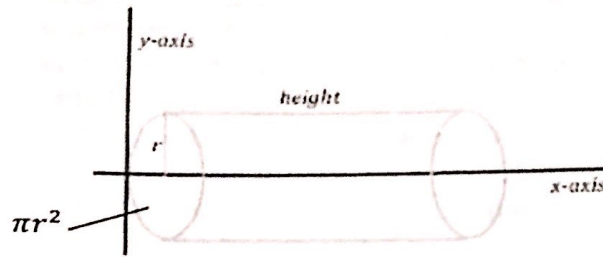
Since the radius of the cylinder didn't change and was equal to 12, I was able to put it in the formula. Again, I contemplated what function to put in for the formula that calculates the total volume of the cone because the radius changes depending on the distance away from the base of the cylinder. I had to refer back to the calculations of areas I did previously, and remembered that the radius of the cone at a given height is equal to that height,  $r_2 = y$ . Because the volume of a cone is the sum of all the circle slices, the general function I needed to use was also  $\pi r^2$ . Knowing that, I can replace  $r$  with  $y$ , since the radius of a slice of a cone is equal to the height of the slice distanced from its base. As a result, the new calculation would be:

$$\int_0^{12} \pi(12)^2 dr - \int_0^{12} \pi y^2 dy$$

$$= \left( \frac{144r\pi}{3} r^3 \right) - \left( \frac{\pi}{3} y^3 \right)$$

But I hit a wall again, since  $\left( \frac{144r\pi}{3} r^3 \right)$  did not calculate the volume of the cylinder according to the geometric formula for cylinders lingering in the back of my head. As I realized that I'm not sure whether this equation was correct or workable, I inquired my math teacher. When my teacher explained the concept of integration with a disk, with radius  $r$ , revolving around and extending along the  $x$ -axis with length  $h$ , I was able to further understand how integrals actually work. In discussing how to find the volume of the cone, he drew this diagram, something I had never thought of before:





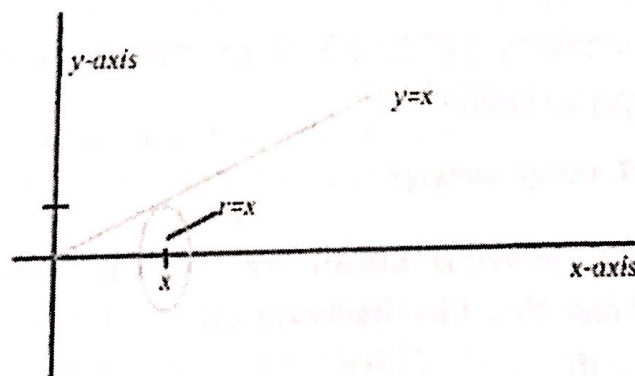
Since the radius and the height of the cylinder, in this case, were equal, the period of which function  $\pi r^2$  operates in would be the height of the cylinder too. With this concept in mind, I was able to comprehend the new equation that my math teacher suggested to find the volume of the cylinder:

$$\int_0^{12} \pi(12)^2 dy$$

The slight modification that was made was changing  $dr$  to  $dy$ , yet it made a huge difference! It is evident that even the smallest mistake or misconception can take me in a completely different direction. Although in this exploration  $r = y$ , I discovered that I cannot simply differentiate with respect to  $r$  because technically  $r$  is not the height of the cylinder. Also, that would give me an equation that doesn't calculate the volume of a cylinder based on my prior knowledge of volume formulas. Interestingly though, when I differentiated with respect to  $y$ , the actual height variable of the cylinder, the equation worked out smoothly with that slight modification:

$$\begin{aligned} \int_0^{12} \pi(12)^2 dy \\ &= 144\pi y \\ &= 144\pi(12) - 144\pi(0) \\ &= 1728\pi \end{aligned}$$

As for finding the volume for a cone, he drew this:



He pointed out the fact that the outline of the cone can actually be represented by the line equation  $y = x$ , something I've never realized before.

This made the process of constructing an integral formula visually concrete for me since it physically showed that  $r = x$ , consequently  $y = x$ . I differentiated with respect to  $y$  again since  $y$  is the actual height of the cone, as in the case of the cylinder. Although the first equation I wrote out was actually correct, here I realized I initially looked at integrals and calculus with a shallow lens, not fully comprehending anything, and now I was able to understand what each part of the formula represent and does, especially in relation to this problem. I proceeded to find the volume of the cone:

$$\begin{aligned} & \int_0^{12} \pi y^2 dy \\ &= \frac{\pi y^3}{3} \\ &= \frac{(12)^3 \pi}{3} - \frac{(0)^3 \pi}{3} \\ &= \frac{1728\pi}{3} \\ &= 576\pi \end{aligned}$$

And to find the volume of cylinder minus a cone:

$$\begin{aligned} &= 1728\pi - 576\pi \\ &= 1152\pi \end{aligned}$$

The volume of the hemisphere was equal to the volume of the cylinder minus a cone! But in order to verify my answers, I used geometric formulas to find volumes to see if I had calculated the above numbers correctly.

*Area of hemisphere:*

$$\begin{aligned} &= \frac{2}{3}\pi r^3 \\ &= \frac{2}{3}\pi(12)^3 \\ &= \frac{3456}{3}\pi \\ &= 1152\pi \end{aligned}$$

*Area of cylinder:*

$$\begin{aligned} &= \pi r^2 h \\ &= \pi(12)^2(12) \\ &= 1728\pi \end{aligned}$$

*Area of cone:*

$$\begin{aligned} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi(12)^2(12) \\ &= \frac{1}{3}(1728)\pi \\ &= \frac{1728}{3}\pi \\ &= 576\pi \end{aligned}$$

Since the volume of both solids equal  $1152\pi$ , I have successfully showed how the Cavalieri's principle may apply to geometry!

### Conclusion

As I was understanding and proving Cavalieri's principle in this exploration, I was impressed by the fact that Cavalieri was able to incorporate calculus, which was not discovered until one century later, into his seemingly simplistic yet complex principle. Cavalieri was truly a man ahead of his time! It was fascinating and interesting to have gone through and understood Cavalieri's



principle in a step-by-step manner, even though there were obstacles along the way and I was often confused.

The biggest lesson I learned after this mathematical exploration was that it is important to have a real understanding of a concept before putting it to use. I went ahead and attempted to do integral calculus with a brief understanding of it and used it by copying the formula and, of course, I was initially unsuccessful. Yet, with thorough explanation and helpful diagrams, my previously limited knowledge of integral calculus expanded as I started to understand how each puzzle of the formula makes the integral formula work. As a result, I finished this exploration with the ability to apply and use the integral formula better in the future without getting confused or lost. I also realized that even the smallest mistake would completely change the result; by using or writing an incorrect notation, the answer I got was entirely different!

It would be another worthy investigation to see how Cavalieri's principle may apply to any other three-dimensional shapes that, at first sight, may not look like they would have equal volume. Perhaps I can even construct one myself. Throughout and after this mathematical exploration, I began to wonder how Cavalieri stumbled upon this simple yet very neat principle. What inspired and caused him to compare the cross sections and the volume of two seemingly dissimilar figures? Also, did Newton and Leibniz, mathematicians in the later years, consult the findings of Cavalieri when they discovered calculus? These are certainly thought-provoking questions that I would like explore further.

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