Examining the Fibonacci Sequence and Its Implications in Nature

Math SL IA

Introduction

This paper aims to explore the Fibonacci Sequence, founded by Leonardo "Fibonacci" Pisano Bigollo in 1202. Fibonacci derived this sequence while studying the growth of a hypothetical problem concerning the population of rabbits. This paper will also aim to validate the sequence, and to examine the frequency that Fibonacci numbers appear in nature, specifically in plants. My interest in this topic was originally sparked when I came across a video on YouTube of a girl talking about the relationship between the Fibonacci sequence and spirals. She then went on to examine the overwhelming presence of spirals and Fibonacci numbers in nature. After watching this video, I was interested in further examining the relationship, and this seemed to be perfect moment to explore it myself.

The Sequence

Fibonacci's Sequence is a sequence of numbers, where the next number of the pattern is found by adding the two previous numbers. The sequence starts with 0, and then proceeds with 1, 1, 2, 3, 5, 8, 13, 21, and so on. To find a specific term in the sequence, one can use the equation:

$$x_n = x_{n-1} + x_{n-2}$$

Where x_n is the term number "n", x_{n-1} is the term before the one you're trying to find, and x_{n-2} is the term before that.

Proving the Sequence

Now I will prove the sequence to validate Fibonacci's findings. To start off the sequence, we begin with 1. Then, due to the fact that the number that comes before that is 0, we add 0 to 1. The resulting number is 1. Now we add 1 to 1 and we get 2. To find the following terms, we just continue the pattern:

0+1=1

1+1=2

1+2=3

2+3=5

3+5=8

5+8= 13

8+13=21

By using induction, due to the fact that all terms are defined in terms of all the smaller numbers before them, we must use first prove that P(1) is true to form a basis for the rest of our proof. Then we can move on to prove that P(n+1) is true. First we let:

$$r = \frac{1+\sqrt{5}}{2} \approx 1.62$$

If this is true, then it fulfills that:

$$r^2 = r + 1$$

And that in turn verifies that:

$$f_n \geq r^{n-2}$$

Now we move on to prove that P(1) is true. While looking at our sequence, we can determine that $f_1 = 1$, but we can also prove this by saying that $r^{1-2} = r^{-1} \leq 1$. Because the second term in the sequence is also one, we can prove this theory farther by saying that $f_2 = 1$, and that $r^{2-2} = r^0 = 1$.

Now that we have established a basis, we must now prove an equation for when $n > 1$ is fixed. The hypothesis from our first step is that P(1), P (2), P(3)…. And P(n) are all true. The next step is to prove P($n + 1$), or that $f_{n+1} \geq r^{n-1}$. To begin, we start with:

$$f_{n+1} = f_n + f_{n-1}$$

And then use our induction hypothesis to determine that $f_n \geq r^{n-2}$ and that $f_{n-1} \geq r^{n-3}$. Substitution can then be applied to come up with:

$$f_{n+1} = r^{n-2} + r^{n-3}$$

If we factor a r^{n-3} out of the equation, it gives us:

$$f_{n+1} = r^{n-3}(r+1)$$

Now we use $r^2 = r + 1$, which we previously found and incorporate that into our equation to find:

$$f_{n+1} = r^{n-3}(r+1) = r^{n-3} * r^2 = r^{n-1}$$

Now we have proven that $f_{n+1} \geq r^{n-1}$, and validated our sequence.

Phi

The number ϕ is also incorporated into the Fibonacci Sequence. The ratio between any two consecutive numbers in the sequence is approximately 1.618034, which is ϕ. This also works with any two random numbers from the sequence. ϕ can also be used to find a term number in the sequence with the equation:

$$x_n = \frac{\Phi^n - (-\Phi)^n}{\sqrt{5}}$$

To prove this, we first must know that $\Phi = \frac{\sqrt{5}+1}{2}$, $\varphi = \frac{\sqrt{5}-1}{2}$, $\Phi - \varphi = 1$, and $\Phi * \varphi = 1$. Also, it is important to know that ϕ and φ are the two roots of $x^2 = x + 1$. This leads to the conclusion that

$x^n = fib(n)x + fib(n-1)$ when $n \geq 0$ and when $fib(n)$ is a number in the Fibonacci Sequence. From this formula, we can derive that:

$$\phi^n = fib(n)\phi + fib(n-1) \text{ and } -\varphi^n = fib(n)(-\varphi) + fib(n-1)$$

If we subtract $-\varphi^n$ from ϕ^n, it gives us:

$$\phi^n - (-\varphi)^n = fib(n)(\phi - (-\varphi))$$

And to solve for $fib(n)$, we divide $\phi - (-\varphi)$ out to get:

$$fib(n) = \frac{\phi^n - (-\varphi)^n}{\phi - (-\varphi)}$$

Now we can refer to the fact that $\phi - (-\varphi) = \sqrt{5}$, which leads to:

$$fib(n) = \frac{\phi^n - (-\varphi)^n}{\sqrt{5}}$$

$\varphi = 1/\phi$, and this means that:

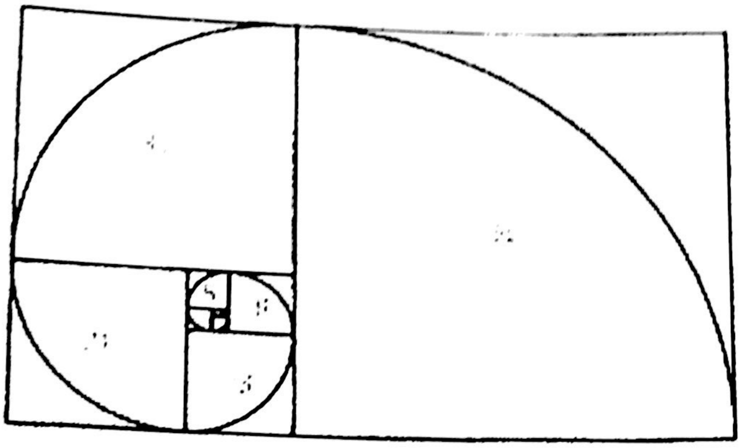
$$(-\varphi)^n = (-1 * \varphi)^n - \left(\frac{1}{(-1-\varphi)}\right)^{-n} = \left(\frac{-1}{\varphi}\right)^{-n} = (-\phi)^{-n}$$

After finding this, we can make our final substitution for our final equation of:

$$fib(n) = \frac{\phi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

Spirals

Another topic related to the Fibonacci Sequence is Fibonacci Spirals. To get a Fibonacci Spiral, you first draw a one by one square. Add another square of the same height next to it. Next you add a bigger, two by two square, that incorporates the first two squares on one of the sides. You then continue to add larger squares incorporating the previous rectangle made as a side. Then, if you draw a spiral connecting through the squares you will get a Fibonacci Spiral (see below). The sum of the squares of any Fibonacci numbers is equal to the last number in the Fibonacci sequence that was used multiplied by the next Fibonacci number. As the spiral progresses, the ration between the numbers gets even closer and closer to ϕ.



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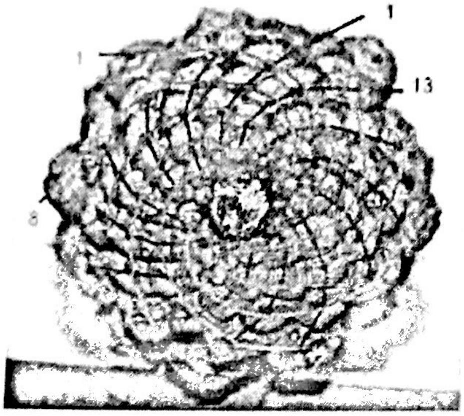
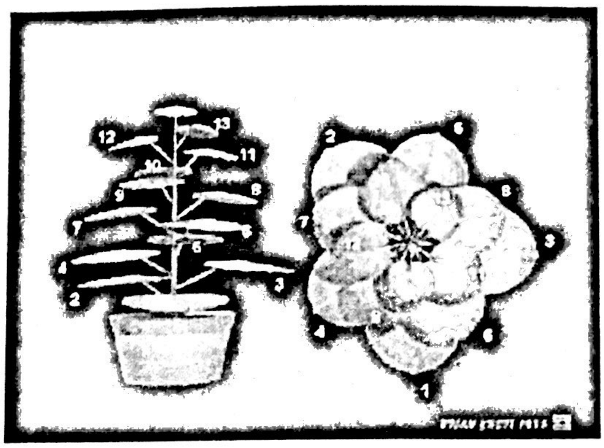
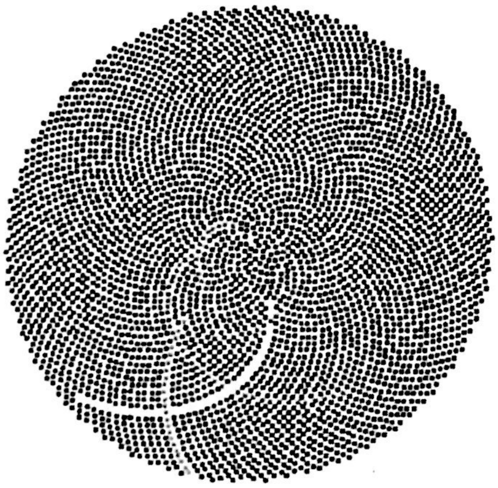
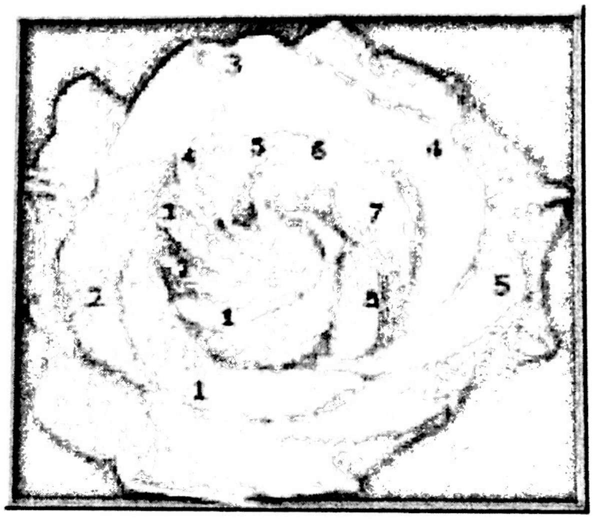
Applying it in Nature

The Fibonacci Sequence is a pretty cool sequence by itself, but its applications in the real world are what really drew my interest in this topic. Fibonacci numbers are apparent everywhere in nature, from pinecones, to petals on leaves, to seashells, to ears, but in this exploration I will specifically be focusing on Fibonacci sequences and numbers in plants. In nature, plants practically apply Fibonacci numbers in the arrangement of their leaves and petals, the number of their petals or leaves, and the arrangement of their seeds. Plants use this sequence to try to space their petals far enough apart to obtain optimal sunlight. If they space them out with any angle that had a whole number at the base of the fraction, there would eventually be overlap and sunlight would be lost, which would lead to a decrease in their ability to photosynthesize. In order to eradicate overlap, the most irrational number is needed to be used, which is ϕ. If a plant spaces out its leaves or petals at this angle, which is approximately 222.5° around the stem from the previous leaf or petal, the leaves would end up spiraling out, but never fully overlap. These numbers appear in the spirals of leaves that go around the stem, where each spiral is the number of leaves counted until you encounter a leaf directly above the starting one. The numbers of turns in each direction and the number of the leaves met almost always are 3 consecutive Fibonacci numbers. Flowers also use Fibonacci Spirals in the packing of their seeds in the seed head. This spiral pattern optimizes the number of seeds that are able to packed, while also keeping the seeds uniformly packed. There is no cluttering at the center of the seed head and seeds are not too spaced out around the outside. With larger flowers, the spirals are larger and farther out, while smaller flowers have smaller and more compact spirals. The flower packs each seed a ϕth of a turn from the last seed and a little further out.









Conclusion

When I originally learned of Fibonacci's sequence, it seemed like quite a simple sequence. You just add the two numbers before to get the next term. But after further exploring the math and the proofs behind sequence, it made me realize how complex this sequence is. Also, through further exploring the applications that this sequence has in the world around it, it served to further spark my interest in math. The abundance that these numbers appear in every aspect of the way plants grow to optimize the amount of sunlight and rain it can receive is truly incredible, but it also demonstrates that math is really applicible in the real world. The way Fibonacci also was able to derive this entire sequence also serves to inspire people in the magic of math. Not only is this sequence practical and applicable everywhere in nature, what seems like a simple sequence that really has more facets as you examine the multiple ways it is aparent in the real world.

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